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THE k-FACTOR CONJECTURE IS TRUE*

Sukhamay KUNDU**

Department of Mathematics and Electronics Research Laboratory, University of California, Berkeley, Calif. 94720, USA

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Abstract. A sequence $\langle d_i \rangle$, $1 \leq i \leq n$, is called graphical if there exists a graph whose i^{th} vertex has degree d_i for all i. It is shown that the sequences $\langle d_i \rangle$ and $\langle d_i - k \rangle$ are graphical only if there exists a graph G whose degree sequence is $\langle d_i \rangle$ and which has a regular subgraph with k lines at each vertex. Similar theorems have been obtained for digraphs. These theorems resolve conjectures given by A.R. Rao and S.B. Rao, and by B. Grünbaum.

0. Introduction

Our aim in this paper is to prove the following two conjectures. The conjectures were made by Rao and Rao [8]. Grünbaum [2] conjectured independently the special case $k = 1$ of Conjecture 0.1.

Conjecture 0.1. *If the sequences $\langle d_i \rangle$ and $\langle d_i - k \rangle$ are graphical, then $\langle d_i \rangle$ can be realized by a graph G which contains a k-factor.*

Conjecture 0.2. *If the sequences $\langle d_i^+, d_i^- \rangle$ and $\langle d_i^+ - k, d_i^- - k \rangle$ are realizable by digraphs, then the former sequence has a realization by a digraph containing a k-factor.*

In fact, we shall prove a slightly more general form of each of the two conjectures.

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** Present address: Department of Computer Sciences, University of Texas, Austin, Texas 78712, USA.

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1. Notations and definitions

We shall assume that graphs (digraphs) have no multiple lines (arcs) and loops, and have a fixed set of vertices $V = \{v_1, v_2, ..., v_n\}$. A line with end points v_i and v_j in a graph is denoted by (v_i, v_j), and so will be denoted the arc from v_i to v_j in a digraph. A digraph may contain both the arcs (v_i, v_j) and (v_j, v_i). It will be convenient to identify a graph (resp., a digraph) with its set of lines (resp., arcs). The degree of v_i in a graph is denoted by d_i. In a digraph, d_i^+ (d_i^-) stands for the outdegree (indegree) of v_i. The sequence $\langle d_i \rangle$ (respectively, $\langle d_i^+, d_i^- \rangle$) is called the degree sequence of the graph (resp., the digraph), and is said to be realized by G.[1] A subgraph F of a graph G, denoted by $F \subseteq G$, has for its lines a subset of the lines of G. A subdigraph is defined in a similar way. Given a sequence $\langle k_i \rangle$, a subgraph F is called a *subfactor* (of G) if the degree of vertex v_i in F is at most k_i for all i; vertex v_i is said to be *saturated* if the degree is equal to k_i, and unsaturated otherwise. F is called a *factor* (more precisely, a $\langle k_i \rangle$-factor) of G if all vertices are saturated. In that case F has degree sequence $\langle k_i \rangle$. These concepts can be extended in a natural way to digraphs where we assume given a sequence of pairs of integers $\langle k_i^+, k_i^- \rangle$, k_i^+ for the outdegree and k_i^- for the indegree of vertex v_i. A vertex v_i is said to be outer (inner) saturated in a subfactor F of digraph G if the outdegree (indegree) of v_i is equal to k_i^+ (k_i^-), etc. The next definition is vital for most part of the paper and will be a major tool in the proofs. Given an ordered pair of graphs $\langle G, H \rangle$, we define an *alternating path* as a sequence $P = [(x_0, x_1), (x_1, x_2), (x_2, x_3), ..., (x_{2t}, x_{2t+1})]$ of distinct lines such that (a) two consecutive lines are incident with a common vertex, and (b) (x_i, x_{i+1}) belongs to $G \setminus H$ or $H \setminus G$ according as i is even or odd.[2] When G and H are two digraphs, we define an *alternating chain* as a sequence of distinct arcs of the form $P = [(x_0, x_1), (x_2, x_1), (x_2, x_3), ..., (x_{2t}, x_{2t+1})]$, where the arcs (x_i, x_{i+1}) belong to $G \setminus H$ and arcs (x_i, x_{i-1}) belong to $H \setminus G$. Note that in an alternating chain the second, fourth, ... arcs are traversed in the reverse direction as one visits the vertices $x_0, x_1, x_2, ...$ in that order. Finally, the number of lines (arcs) in P is said to be the length of P, and denoted by $|P|$.

[1] G is used to denote both a graph and a digraph. However, no confusion is likely to happen.
[2] $G \setminus H$ denotes the set of lines (arcs) belonging to G but not H.

The paper is devided into two sections. Undirected graphs are considered in Section 2, and Section 3 is devoted to directed graphs.

2. Undirected graphs

In this section, we shall prove a slightly generalized form of Conjecture 0.1, where we allow two k_i's differ by at most one. Conjecture 0.1 is the special case $k_i = k$ for all i.

Theorem 2.1. *Let $\langle d_i\rangle$ and $\langle d_i-k_i\rangle$ be two graphical sequences with the property that for some $k \geq 0$, $k \leq k_i \leq k+1$ for all i. Then there exists a graph with degree sequence $\langle d_i\rangle$ containing a $\langle k_i\rangle$-factor.*

The main steps in the proof by contradiction are as follows. We choose realizing graphs G and H with degree sequences, respectively, $\langle d_i\rangle$ and $\langle d_i-k_i\rangle$ and a subfactor F in G which is disjoint from H (i.e., $F \subseteq G\backslash H$). First, we observe that if F is not a factor, then an alternating path must exist whose end points are unsaturated. The next step is to show that this path can be used to construct new graphs G, H and F which give a larger value of $m = |F| + |G \cap H|$. The various constructions of G, H and F depending on the nature of the alternating path are hidden in the six-part proof of the theorem. The conclusion is that when m is maximum, F is a factor.

Proof of Theorem 2.1. To simplify the matter, let G and H be a realizing graph of $\langle d_i\rangle$ and $\langle d_i-k_i\rangle$, respectively, and F be a $\langle k_i\rangle$-subfactor contained in $G\backslash H$ such that $|G \cap H| + |F| = m$ is maximum. We can assume that F is not a factor. Let $S = S(F)$ denote the set of unsaturated vertices. If vertex $v_i \notin S$, then it is not hard to see that there exists an alternating path P with respect to the pair of graphs $\langle G\backslash F, H\rangle$ joining vertex v_i to some vertex (possibly same as v_i) in $V\backslash S$. Let $P = [(x_0, x_1), (x_1, x_2), \ldots, (x_{2t}, x_{2t+1})]$, where $x_0 = v_i$, be such a path of shortest length. A contradiction is derived by showing that $|P|$ is both less than 7 and greater than 5. It is important to note that vertices x_1 and x_{2t} are necessarily saturated (since F is a largest subfactor in $G\backslash H$). The proof has six parts 1–6 in all.

Part 1. *If $(y_0, y_1) \in H\backslash G$, $(y_1, y_2) \in G\backslash H$ but is not in F, and $(y_2, y_3) \in H\backslash G$, then $(y_0, y_3) \in H\backslash G$.*

Proof. If we had $(y_0, y_3) \in G$, we could delete it and (y_1, y_2) from G, and introduce (y_0, y_1) and (y_2, y_3) as lines of G. If (y_0, y_3) were in F, we would also delete it from F. In both cases, the value of m is increased. If we had $(y_0, y_3) \notin H$, then we could [illegible]) and (y_2, y_3) from H and add the lines (y_0, y_3) and $(y_1,$ [illegible] an increasing m.

Part 2. *Let* $Q = [(y_0, y_1), (y_1, y_2), ..., (y_{2q}, y_{2q+1})]$, $q \geq 1$, *be an alternating path with respect to the ordered pair* $\langle G, H \rangle$ *which is disjoint from* F. *If* $y_0 \neq y_{2q+1}$ *and* $(y_0, y_{2q+1}) \notin Q$, *then* (y_0, y_{2q+1}) *belongs to* $G \backslash H$.

Proof. The proof is similar to that of Part 1 and will be omitted.

It will be shown in Part 6 (being postponed for reasons that will become clear later) that the shortest alternating path P has at most five lines. Moreover, if the length of P is five, then $x_1 = x_4$ (see Fig. 1).[3] We shall show that both $|P| = 3$ and $|P| = 5$ contradict the maximality of m. In the case $|P| = 3$, there are two possibilities: $x_0 \neq x_3$ (see Fig. 2) and $x_0 = x_3$. The more difficult case $x_0 = x_3$ will be considered in Part 4 and 5, while Part 3 deals with the other two cases.

Part 3. *Each of the following two assumptions contradicts the maximality of m. The shortest alternating path P has*

(i) *5 lines,*

(ii) *3 lines and distinct end points.*

Proof. Let $y = x_5$ or x_3 according as we are in case (i) or (ii). In view of Part 2, $(x_0, y) \in G \backslash H$, and moreover, the vertices x_0 and y being unsaturated, we have $(x_0, y) \in F$. Thus $k(y) \geq 2$, where $k(y) = k_i$ if $y = v_i$, which in turn implies that $k(x_1) \geq 1$. Recall that x_1 is saturated and let $(x_1, u) \in F$. In case (ii), we can assume that $u \neq y$. For otherwise $k(y) \geq 3$, and hence $k(x_1)$ being at least 2, there exists a vertex $v \neq x_3$ such that $(v, x_1) \in F$. Now we define a subfactor $F' = F + (x_0, x_1) - (u, x_1)$; $|F| = |F'|$ and the vertices u and y are unsaturated in F'. Replacing (x_0, x_1) by (u, x_1) in P, we get an alternating path which is disjoint from F'. We conclude as above that $(u, y) \in F'$, or equivalently $(u, y) \in F$. Repeated application of the above argument leads to $k(y) \geq n$ which is impossible.

[3] In Fig. 1–4 the letters α and β indicate, respectively, that the line belongs to $G \backslash H$ and $H \backslash G$.

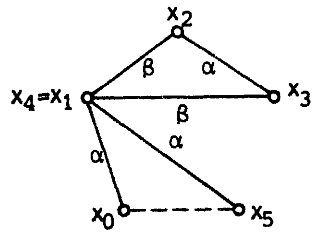

Fig. 1. The shortest alternating path $P = [(x_0, x_1), (x_1, x_2), \ldots, (x_4, x_5)]$ of length five. The broken line (x_0, x_5) belongs to F.

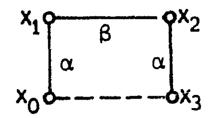

Fig. 2. $|P| = 3$ and $x_0 \neq x_3$. The line (x_0, x_3) belongs to F.

Part 4. *Suppose $|P| = 3$ and $x_0 = x_3$. Then there exists at least one line of H incident with the vertex x_0 which is not in G.*

Proof. Otherwise $k(x_0)$ = degree of x_0 in $G \setminus H$ and $k(x_1) \geq k(x_0) - 1 \geq 1$. Let $(u, x_1) \in F$, and F' be defined as in the proof of Part 3. However, with respect to the alternating path $Q = [(u, x_1), (x_1, x_2), (x_2, x_3)]$ we are back to Part 3(ii) which is shown to give a contradiction. The assertion is proved.

We next show that Part 4 cannot possibly be true either.

Part 5. *If $|P| = 3$ and $x_0 = x_3$, then the degree of x_0 in $H \setminus G$ is zero.*

Proof. If possible, let there be a line incident with x_0 which is in H but not G. Consider all alternating paths Q with respect to the pair $\langle G \setminus F, H \rangle$ which contains the path P properly as an initial part and terminates in the set $V \setminus S$. It is easy to show that such a path, indeed, exists. Clearly, a path Q cannot return to $x_0 = x_3$ after (x_2, x_3). Let $Q = P \cup P_0$ be the shortest alternating path containing P. First, we show that $|P_0| = 2$ (see Fig. 3). Suppose that $|P_0| \geq 3$ and let $(x_3, x_4), (x_4, x_5), (x_5, x_6)$ be the first three lines of P_0. In view of Part 1 and the alternating paths

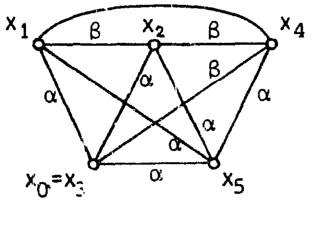

Fig. 3. To show that x_0 has degree zero in $H \setminus G$. $P_0 = [(x_3, x_4), (x_4, x_5)]$.

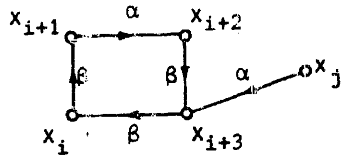

Fig. 4.

$[(x_1, x_2), (x_2, x_3), (x_3, x_4)]$ and $[(x_2, x_1), (x_1, x_3), (x_3, x_4)]$, we get (x_4, x_1) and (x_4, x_2) belong to $H\backslash G$. Similarly, one can show using Part 2 that $(x_5, x_j) \in G\backslash H$ for $1 \leq j \leq 3$. It follows that the vertices $\{x_j : 1 \leq j \leq 6\}$ are distinct, and (x_3, x_6) which is in $H\backslash G$ due to Part 1 does not belong to Q. Now we can replace the first three lines of P_0 by the single line (x_3, x_6) and obtain a shorter alternating path Q', a contradiction. Thus P_0 has only two lines. But then the alternating path $[(x_0, x_1), (x_1, x_4), (x_4, x_5)]$ and the assertion 3(ii) are in contradiction with each other. This completes the proof of Part 5.

The assertions 3–5 together with a proof of $|P| \leq 5$ will complete the proof of the theorem. As our final step in the proof we show that:

Part 6. *If $P = [(x_0, x_1), (x_1, x_2), ..., (x_{2t}, x_{2t+1})]$ is the shortest alternating path from $x_0 = v_i$ to an unsaturated vertex x_{2t+1} and x_0 itself is unsaturated, then P has at most 5 lines.*

Proof. It suffices to show that two successive lines of P belonging to H are incident with a common vertex. We shall regard for the moment that the lines of P are oriented in the direction x_i to x_{i+1}. In order to stress the orientation we sometimes write them as $(x_i, x_{i+1})^-$ (read: arc (x_i, x_{i+1})). First, we observe that at each vertex v_j the path P may not have more than one line from each of H and G directed from v_j. For example, if there were two lines of H directed from v_j, then one of them preceded the other in the sequence P, and the subsequence between them would contain an alternating cycle of even length, contradicting the maximality of m. Similar remarks hold for the lines of P directed into v_j. Suppose now that (x_i, x_{i+1}) and (x_{i+2}, x_{i+3}) are two consecutive lines of $P \cap H$ which do not have a common vertex, i.e., $x_i \neq x_{i+3}$. It follows first of all that (x_i, x_{i+3}) belongs to $H\backslash G$, and then (x_i, x_{i+3}) must be in P (otherwise, we could replace the lines (x_i, x_{i+1}), (x_{i+1}, x_{i+2}),

(x_{i+2}, x_{i+3}) by the line (x_i, x_{i+3}) in P giving a shorter alternating path) and be oriented from x_{i+3} into x_i (see Fig. 4). Let (x_{j-1}, x_j) and $(x_j, x_{j+1} = x_{i+3})$ be the two lines of P immediately preceding (x_{i+3}, x_i); the line (x_{j-1}, x_j) exists if $x_j \neq x_0$. Note that the vertices x_j, x_{j-1} and x_p, where $i \leq p \leq i+3$, are distinct. In view of Part 1, (x_i, x_{j-1}) belongs to $H \setminus G$ and it is not in the path P. But then there is a shorter alternating path than P obtained by replacing the subsequence $[(x_{j-1}, x_j), (x_j, x_{i+3}), (x_{i+3}, x_i)]$ by (x_{j-1}, x_i). Therefore $x_j = x_0$. Likewise, the line following $(x_{i+3}, x_i)^-$ must be $(x_i, x_{2t+1})^-$. Now we have the following alternating path $[(x_0, x_{i+3}), (x_{i+3}, x_i), (x_i, x_{2t+1})]$, a contradiction. Hence $x_i = x_{i+3}$.

Remark 2.2. (a) Since the existence of a $\langle k_i\rangle$-factor implies the existence of a $\langle d_i - k_i\rangle$-factor, and vice-versa, it is enough to assume that the inequalities $k \leq f_i \leq k+1$ hold for $f_i = k_i$ or $f_i = d_i - k_i$ and all i. Moreover, by considering the complement graphs, one can show that Theorem 2.1 holds true if the inequalities $k \leq d_i \leq k+1$ are valid and k_i's are arbitrary. Thus it suffices to assume that one of the sequences $\langle d_i\rangle$, $\langle k_i\rangle$ and $\langle d_i - k_i\rangle$ satisfy the almost regularity condition.

(b) To construct the required graphs G and F one may proceed as follows. Start with arbitrary realizing graphs G and H for the sequences $\langle d_i\rangle$ and $\langle d_i - k_i\rangle$, respectively. Choose any subfactor F contained in G but disjoint from H which is maximal as a set of lines. If F is not a factor, then an alternating path joining two unsaturated vertices may be obtained. Now use one of the various constructions indicated in the proofs of 1–6 to increase the current value of $|F| + |G \cap H|$. Repeat the last two steps until F is a factor.

The following example shows that the theorem may not be true if the maximum and minimum of k_i, $1 \leq i \leq n$, differ by two or more.

Example 2.3. Let $\langle d_i\rangle = \langle 5,5,4,3,3,2\rangle$ and $\langle k_i\rangle = \langle 3,3,1,3,3,1\rangle$. Both the sequences are graphical. It is easy to see that there is no realization of $\langle d_i\rangle$ containing the (unique) realizing graph for $\langle d_i - k_i\rangle$. Another example would be the sequences $\langle 4,3,2,2,2,1\rangle$ and $\langle 1,2,0,0,1\rangle$.

The following corollary is a direct consequence of Theorem 2.1 and a theorem of Rao and Rao [7,8] on connected factors.

Corollary 2.4. *Let $\langle d_i \rangle$ be a non-increasing sequence. Then there exists a realization of $\langle d_i \rangle$ containing a hamiltonian cycle if and only if*

(i) *the sequences $\langle d_i \rangle$ and $\langle d_i - 2 \rangle$ are graphical, and*

(ii) *the following inequality holds for all $p < \frac{1}{2}n$:*

$$\sum_{i \le p} d_i < p(n-p-1) + \sum_{i \ge n-p+1} d_i .$$

Our next corollary is an extension of a theorem of Rao and Rao.

Corollary 2.5. *If there exists a realizing graph of $\langle d_i \rangle$ containing a k-factor, then for $0 < p < k$, where $p = k$ (mod 2) if n is odd, there exists a realizing graph containing a p-factor.*

Proof. It suffices to show that $\langle d_i - p \rangle$ is graphical. However, this follows from some symmetrization (as observed by A.R. Rao) of a theorem of Fulkerson [1] on the existence of (0,1)-matrices with given row sums and column sums.

The case $p = k$ (mod 2) and arbitrary n was obtained previously by Rao and Rao [7].

3. Directed graphs

In this section, we prove a generalized form of the second conjecture. The generalization consists of choosing only one of the two sequences $\langle k_i^+ \rangle$ and $\langle k_i^- \rangle$ constant, and leave the other one arbitrary.

Theorem 3.1. *Let $\langle d_i^+, d_i^- \rangle$ be a graphical sequence and $\langle k_i^+, k_i^- \rangle$ have the property that for some $k \ge 0$, $k = k_i^-$ (or, $k = k_i^+$ for that matter) for all i. Then there exists a realizing digraph of $\langle d_i^+, d_i^- \rangle$ containing a $\langle k_i^+, k_i^- \rangle$-factor if and only if the sequence $\langle d_i^+ - k_i^+, d_i^- - k_i^- \rangle$ is graphical.*

That the sequence $\langle d_i^+ - k_i^+, d_i^- - k_i^- \rangle$ be graphical is clearly necessary. The essence of the theorem is the fact that it is also sufficient. Conjecture 0.2 is the special case of Theorem 3.1, where $k_i^+ = k_i^- = k$ for all i. The main idea in the proof of Theorem 3.1 by contradiction is very similar to the one used in proving Theorem 2.1. Now the alternating chains enter into the game replacing the alternating paths. It so happens that the proof is somewhat less involved in the case of digraphs.

Proof of Theorem 3.1. Let G and H be two digraphs with degree sequences $\langle d_i^+, d_i^-\rangle$ and $\langle d_i^+ - k_i^+, d_i^- - k_i^-\rangle$, respectively, and let F be a $\langle k_i^+, k_i^-\rangle$ subfactor contained in $G\backslash H$. We shall assume that the digraphs G, H and F are so chosen that $m = |F| + |G \cap H|$ is maximum and prove that F is a factor. Suppose that F is not a factor, and let v_i be a vertex which is not outer saturated. It is not hard to see that there exists an alternating chain $P = [(x_0, x_1), (x_2, x_1), (x_2, x_3), \ldots, (x_{2t}, x_{2t+1})]$ with respect to the pair of digraphs $\langle G\backslash F, H\rangle$ whose initial vertex is $v_i = x_0$ and the final vertex x_{2t+1} is inner unsaturated. There are two cases to consider: (a) $x_0 \neq x_{2t+1}$, and (b) $x_0 = x_{2t+1}$. We first observe that if $x_0 \neq x_{2t+1}$, then $(x_0, x_{2t+1}) \notin P$. This is proved as follows. Assume the contrary, and let (x_0, x_{2t+1}) be the j^{th} arc in P. If $(x_0, x_{2t+1}) \in H$, then the first j arcs of P form a closed alternating chain. On the other hand if $(x_0, x_{2t+1}) \in G$, we get the first $(j-1)$ arcs of P form a closed alternating chain. But how can at all such a chain exist if m is maximum! If we let P' denote the closed chain, then we could delete the arcs $G \cap P'$ from G and introduce the arcs $H \cap P'$ to G, increasing the value of m. Thus $(x_0, x_{2t+1}) \notin P$ and, for similar reasons, it must belong to $G\backslash H$. Consequently, $(x_0, x_{2t+1}) \in F$. Now we go back to the consideration of the cases (a) and (b).

(a) Suppose $x_0 \neq x_{2t+1}$. Since $k^-(x_1) = k^-(x_{2t+1}) \geq 2$, where $k^-(y) = k_i^-$ if $y = v_i$, there exists a vertex u such that $(u, x_1) \in F$, and $u \neq x_{2t+1}$. We form the subfactor $F' = F + (x_0, x_1) - (u, x_1)$; $|F| = |F'|$. Replacing the arc (x_0, x_1) by (u, x_1) in P we get an alternating chain with respect to $\langle G\backslash F', H\rangle$. This implies that $(u, x_{2t+1}) \in F'$ and hence it belongs to F. But this being true for all u, we get $k^-(x_{2t+1}) \geq k^-(x_1) + 1$ (discounting one for the possibility that $(x_{2t+1}, x_1) \in F$) which is a contradiction.

(b) Suppose $x_0 = x_{2t+1}$. Then $k^-(x_1) = k^-(x_{2t+1}) \geq 1$. As in case (a), one can show that for every vertex u, where $(u, x_1) \in F$, the arc (u, x_{2t+1}) is in F. Once again we get $k^-(x_{2t+1}) \geq k^-(x_1) + 1$. The theorem is proved.

Remarks similar to those following Theorem 2.1 hold true for the digraphs. In particular, Theorem 3.1 is true if any of $\langle d_i^+\rangle$, $\langle k_i^+\rangle$, $\langle d_i^+ - k_i^+\rangle$, or the corresponding sequences of indegrees is constant. Example 3.2 shows that Theorem 3.1 is sharp in the sense that there are sequences where none of the indegree or outdegree sequences are constant and for which the conclusion is false.

Example 3.2. The following are three sequences that are realizable by digraphs. The outdegrees in the second sequence vary only by 1. Yet there is no digraph whose degree sequence is the first sequence and which contains a factor given by the second sequence.

$$\langle(4,4), (3,3), (2,2), (2,2), (1,1)\rangle ,$$
$$\langle(1,1), (1,2), (1,0), (0,0), (1,1)\rangle ,$$
$$\langle(3,3), (2,1), (1,2), (2,2), (0,0)\rangle .$$

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Note added. Kleitman and Wang [3] have recently generalized both Theorems 2.1 and 3.1, and have given constructive proofs in each case. A very short and elegant proof of Conjecture 0.1 for $k = 1$ was found by Lovász [6]. Also V. Chungphaisan has proved independently the case $k = 1$ and 2 of Conjecture 0.1.

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